

Finsleroid-Finsler Space of Involutive Case

G.S. Asanov

*Division of Theoretical Physics, Moscow State University
119992 Moscow, Russia
(e-mail: asanov@newmail.ru)*

Abstract

The Finsleroid-Finsler space is constructed over an underlying Riemannian space by the help of a scalar $g(x)$ and an input 1-form b of unit length. Explicit form of the entailed tensors, as well as the respective spray coefficients, is evaluated. The involutive case means the framework in which the characteristic scalar $g(x)$ may vary in the direction assigned by b , such that $dg = \mu b$ with a scalar $\mu(x)$. We show by required calculation that the involutive case realizes through the A -special relation the picture that instead of the Landsberg condition $\dot{A}_{ijk} = 0$ we have the vanishing $\dot{\alpha}_{ijk} = 0$ with the normalized tensor $\alpha_{ijk} = A_{ijk}/||A||$. Under the involutive condition, the derivative tensor $A_{i|j}$ and the curvature tensor R^i_k have explicitly been found, assuming the input 1-form b be parallel.

Key words: Finsler metrics, spray coefficients, curvature tensors.

1. Introduction and synopsis of new conclusions

Among various possible methods to specify the Finsler space, raising forth the Landsberg condition $\dot{A}_{ijk} = 0$ occupies an important geometrical role (see [1-3]). In the Finsleroid–Finsler space, the condition can be realized in a simple and attractive way [4-7]. At the same time, the condition requires the Finsleroid charge g to be a constant. How should we overcome the restriction?

At the first sight, in the Finsler geometry the weak Landsberg condition $\dot{A}_i = 0$ is to be considered as being a next-step extension of the proper Landsberg condition $\dot{A}_{ijk} = 0$. However, in the Finsleroid–Finsler space both the conditions are tantamount (because of the particular representation (1.13) of the Cartan tensor A_{ijk}).

A scrupulous analysis performed has revealed a remarkable observation that an attractive method to permit $g \neq \text{const}$ is to use the nullification condition $\dot{\alpha}_{ijk} = 0$ with the normalized Cartan tensor α_{ijk} (which is defined by (1.18)). Clearly, the condition is attained when the A -special relation (2.5) holds. Remarkably, the relation occurs being reachable upon assuming that the scalar $g(x)$ reveals the involutive behaviour: $dg = \mu(x)b$ (see (3.1)).

In Section 2 we indicate the interesting implications of the A -special condition, including the observation that the skew-part of the hv -curvature tensor is proportional to the indicatrix curvature tensor (according to (2.15)). It is the part that enters the right-hand side of the covariant conservation law (2.20).

In Section 3 the involutive case is formulated, showing that under the b -parallel condition (which reads $\nabla b = 0$, where ∇ means the Riemannian covariant derivative operative in the associated Riemannian space) the case entails the A -special relation.

In Conclusions, several important ideas motivated our approach are emphasized.

Clear explicit representations of the basic tensors involved are obtained systematically in Appendix A by means of direct calculation.

In Appendix B, we indicate the explicit form for the spray coefficients of the space under study. They include the part E^k which involves the gradient of $g(x)$ (see (B.20) and (B.21)).

Appendix C is devoted to evaluations in the involutive case. Two key tensors have been explicitly evaluated, namely, $A_{i|j}$ (given by (C.19)) and R^i_k (given by (C.29)), assuming that the 1-form b is parallel (such that $\nabla b = 0$).

We deal with the Finsler space notion which is specified by the condition that the Finslerian metric function $K(x, y)$ be of the functional dependence

$$K(x, y) = \Phi(g(x), b_i(x), a_{ij}(x), y) \quad (1.1)$$

of the particular case given by the formulas (A.36)–(A.41) of Appendix A. In (1.1), the argument set $(g(x), b_i(x), a_{ij}(x))$ involves, respectively, a scalar, a covariant vector field, and a Riemannian metric tensor.

The Finsleroid–Finsler space can be constructed as follows. Let M be an N -dimensional C^∞ differentiable manifold, $T_x M$ denote the tangent space to M at a point $x \in M$, and $y \in T_x M \setminus 0$ mean tangent vectors. Suppose we are given on M a positive-definite Riemannian metric

$$\mathcal{S} = S(x, y).$$

Denote by

$$\mathcal{R}_N = (M, \mathcal{S})$$

the obtained N -dimensional Riemannian space. Let us also assume that the manifold M admits a non-vanishing 1-form

$$b = b(x, y)$$

and

$$||b||_{\text{Riemannian}} = 1. \quad (1.2)$$

It is convenient to use the variable

$$q = \sqrt{S^2 - b^2}. \quad (1.3)$$

The space \mathcal{R}_N entering the above definition is called the *associated Riemannian space*. With respect to natural local coordinates in the space \mathcal{R}_N we have the local representations

$$a^{ij}b_ib_j = 1 \quad (1.4)$$

and

$$b = b_i(x)y^i, \quad (1.5)$$

together with

$$S = \sqrt{a_{ij}(x)y^iy^j}. \quad (1.6)$$

The reciprocity $a^{in}a_{nj} = \delta^i_j$ is assumed, where δ^i_j stands for the Kronecker symbol. The covariant index of the vector b_i will be raised by means of the Riemannian rule

$$b^i = a^{ij}b_j,$$

which inverse reads

$$b_i = a_{ij}b^j.$$

We also introduce the tensor

$$r_{ij}(x) := a_{ij}(x) - b_i(x)b_j(x) \quad (1.7)$$

to have the representation

$$q = \sqrt{r_{ij}(x)y^iy^j}. \quad (1.8)$$

From (1.4) and (1.7) it follows that

$$r_{ij}b^j = 0 \quad (1.9)$$

From the fundamental metric function K , we explicitly calculate distinguished Finslerian tensors, including the covariant tangent vector $\hat{y} = \{y_i\}$, the Finslerian metric tensor $\{g_{ij}\}$ together with the contravariant tensor $\{g^{ij}\}$ defined by the reciprocity conditions $g_{ij}g^{jk} = \delta_i^k$, and the angular metric tensor $\{h_{ij}\}$, by making use of the following conventional Finslerian rules in succession:

$$y_i := \frac{1}{2} \frac{\partial K^2}{\partial y^i}, \quad g_{ij} := \frac{1}{2} \frac{\partial^2 K^2}{\partial y^i \partial y^j} = \frac{\partial y_i}{\partial y^j}, \quad h_{ij} := g_{ij} - l_i l_j, \quad (1.10)$$

where

$$l_i = g_{ij}l^j, \quad l^j = \frac{y^j}{K}. \quad (1.11)$$

After that, we can elucidate the algebraic structure of the associated Cartan tensor

$$A_{ijk} := \frac{K}{2} \frac{\partial g_{ij}}{\partial y^k}, \quad (1.12)$$

which leads to the simple representation

$$A_{ijk} = \frac{1}{N} \left(h_{ij}A_k + h_{ik}A_j + h_{jk}A_i - \frac{1}{A_h A^h} A_i A_j A_k \right) \quad (1.13)$$

(see (A.87)) with

$$A_k = g^{ij} A_{ijk} \quad (1.14)$$

and

$$A_h A^h = \frac{N^2}{4} g^2 \quad (1.15)$$

(see (A.63)).

Owing to (1.15), the norm

$$||A|| = \sqrt{A^k A_k} \quad (1.16)$$

is equal to

$$||A|| = \frac{N}{2} |g(x)|. \quad (1.17)$$

It is convenient to construct the *normalized Cartan tensor*

$$\alpha_{ijk} := \frac{1}{||A||} A_{ijk} \quad (1.18)$$

and the vector

$$\alpha_k := \frac{1}{||A||} A_k \quad (1.19)$$

which length is 1:

$$\alpha_h \alpha^h = 1. \quad (1.20)$$

We have

$$\alpha_{ijk} = \frac{1}{N} (h_{ij}\alpha_k + h_{ik}\alpha_j + h_{jk}\alpha_i - \alpha_i \alpha_j \alpha_k) \quad (1.21)$$

everywhere in the Finsleroid–Finsler space.

In our analysis, an important role is played by the tensor

$$\mathcal{H}_{ij} = h_{ij} - \alpha_i \alpha_j, \quad (1.22)$$

which obviously possesses the nullification properties

$$\mathcal{H}_{ij} y^j = 0, \quad \mathcal{H}_{ij} A^j = 0. \quad (1.23)$$

The curvature of indicatrix is well-known to be described by the tensor

$$\hat{R}_i{}^j{}_{mn} := \frac{1}{K^2} \left(A_h{}^j{}_m A_i{}^h{}_n - A_h{}^j{}_n A_i{}^h{}_m \right). \quad (1.24)$$

In the Finsleroid–Finsler space, the tensor possesses the representation

$$K^2 \hat{R}_{ijmn} = \frac{1}{N^2} (A^k A_k) \left(h_{in} h_{mj} - h_{im} h_{nj} \right) \quad (1.25)$$

(see (A.93)).

In the next section we shall set forth an interesting special condition.

2. A -special condition

By means of the over-dot we denote the action of the operator $|_m l^m$, such that

$$\dot{A}_i = A_{i|m} l^m, \quad \dot{A}_{ijk} = A_{ijk|m} l^m, \quad \dot{\alpha}_i = \alpha_{i|m} l^m, \quad \dot{\alpha}_{ijk} = \alpha_{ijk|m} l^m, \quad (2.1)$$

with $|_m$ meaning the h -covariant derivative (see (B.9)).

Let us set forth the nullification

$$\dot{\alpha}_{ijk} = 0. \quad (2.2)$$

Whenever the representation (1.21) is valid, the condition (2.2) is equivalent to the vanishing

$$\dot{\alpha}_i = 0 \quad (2.3)$$

of the normalized vector (1.19).

Denote

$$\gamma_k = \frac{1}{2A^h A_h} (A^m A_m)_{|k}, \quad \gamma = \frac{1}{2A^h A_h} (A^m A_m)_{|k} l^k. \quad (2.4)$$

Assume that the A -special relation

$$A_{i|k} = \gamma_k A_i + \eta \mathcal{H}_{ik} \quad (2.5)$$

holds, where η is a scalar. The relation (2.5) can obviously be written as

$$\alpha_{i|k} = \eta \mathcal{H}_{ik}. \quad (2.6)$$

Since $\mathcal{H}_{ik} y^k = 0$, from (2.5) we directly conclude that

$$\dot{A}_i = \gamma A_i, \quad (2.7)$$

which is obviously *tantamount to* (2.3). From the representation (1.13) of the tensor A_{ijk} we obtain

$$A_{ijk|l} = \gamma_l A_{ijk} + \eta \frac{1}{N} (\mathcal{H}_{ij} \mathcal{H}_{kl} + \mathcal{H}_{ik} \mathcal{H}_{jl} + \mathcal{H}_{jk} \mathcal{H}_{il}), \quad (2.8)$$

which entails

$$\dot{A}_{ijk} = \gamma A_{ijk}. \quad (2.9)$$

Also, (2.3) entails the nullification

$$\dot{\mathcal{H}}_{jk} = 0 \quad (2.10)$$

(consider the representation (1.22) of the tensor \mathcal{H}_{jk} and take into account that

$$h_{jk|l} = 0$$

in any Finsler space), where

$$\dot{\mathcal{H}}_{jk} = \mathcal{H}_{jk|m} l^m.$$

The hv -curvature tensor

$$P_{jikl} := -(A_{ijl|k} - A_{jkl|i} + A_{kil|j}) + A_{ij}{}^u \dot{A}_{ukl} - A_{jk}{}^u \dot{A}_{uil} + A_{ki}{}^u \dot{A}_{ujl} \quad (2.11)$$

(this representation is tantamount to the definition (3.4.11) on p. 56 of the book [2]) gets reduced upon substituting (2.8) and (2.9):

$$P_{jikl} = -2\eta \frac{1}{N} (\mathcal{H}_{ij}\mathcal{H}_{kl} + \mathcal{H}_{ik}\mathcal{H}_{jl} + \mathcal{H}_{jk}\mathcal{H}_{il}) + \gamma(A_{ij}{}^u A_{ukl} - A_{jk}{}^u A_{uil} + A_{ki}{}^u A_{ujl}). \quad (2.12)$$

Let us consider the skew-part

$$P^{[ji]}{}_{kl} := \frac{1}{2}(P^{ji}{}_{kl} - P^{ij}{}_{kl}). \quad (2.13)$$

From (2.12) it follows that

$$P_{[ji]kl} = \gamma(A_{ki}{}^u A_{ujl} - A_{jk}{}^u A_{uil}). \quad (2.14)$$

In view of the representation (1.24) of the indicatrix curvature tensor $\hat{R}_i{}^j{}_{mn}$, we may write (2.14) as

$$P_{[ji]kl} = \gamma K^2 \hat{R}_{jikl}. \quad (2.15)$$

Thus the following assertion is valid.

THEOREM 2.1. *If the A-special relation (2.5) holds together with the representation (1.13) of the Cartan tensor, then the skew-part of the hv-curvature tensor is proportional to the indicatrix curvature tensor, according to (2.15).*

Since

$$g^{jl} \left(R_j{}^i{}_{il|t} + R_j{}^i{}_{lt|i} + R_j{}^i{}_{ti|l} \right) = P^{li}{}_{iu} R^u{}_{lt} + P^{li}{}_{lu} R^u{}_{ti} + P^{li}{}_{tu} R^u{}_{il} \quad (2.16)$$

(see the formula (3.5.3) on p. 58 of the book [2]), we have

$$g^{jl} \left(R_j{}^i{}_{il|t} + R_j{}^i{}_{lt|i} + R_j{}^i{}_{ti|l} \right) = 2P^{[li]}{}_{iu} R^u{}_{lt} - P^{[li]}{}_{tu} R^u{}_{li}. \quad (2.17)$$

so that the covariant divergence of the tensor

$$\rho_{ij} := \frac{1}{2}(R_i{}^m{}_{mj} + R^m{}_{ijm}) - \frac{1}{2}g_{ij}R^{mn}{}_{nm} \quad (2.18)$$

is given by

$$\rho^i{}_{j|i} = -P^{[lm]}{}_{mu} R^u{}_{lj} + \frac{1}{2}P^{[lm]}{}_{ju} R^u{}_{lm} \quad (2.19)$$

which can be written as

$$\rho^i{}_{j|i} = J_j \quad (2.20)$$

with

$$J_j = P^{[lm]}{}_{ku} \left(-R^u{}_{lj} \delta^k{}_m + \frac{1}{2} R^u{}_{lm} \delta^k{}_j \right). \quad (2.21)$$

Using (2.15) together with (1.25) entails

$$J_j = \frac{1}{4} g^2 \gamma \left(h^l{}_u h^m{}_k - h^l{}_k h^m{}_u \right) \left(-R^u{}_{lj} \delta^k{}_m + \frac{1}{2} R^u{}_{lm} \delta^k{}_j \right). \quad (2.22)$$

3. Finsleroid–Finsler space upon involution

Let us set forth the *involution condition*

$$g_i = \mu b_i, \quad \mu = \mu(x), \quad (3.1)$$

where $g_i = \partial g / \partial x^i$, and formulate the following definition.

Definition. The arisen space

$$\mathcal{IF}\mathcal{F}_g^{PD} := \{\mathcal{F}\mathcal{F}_g^{PD} \text{ with } g_i = \mu b_i, \mu = \mu(x)\} \quad (3.2)$$

is called the *involution Finsleroid–Finsler space*. The involved $\mu(x)$ is called the *involution scalar*.

In the space (3.2), the quantities defined in (2.4) become simply

$$\gamma_k = \frac{1}{g} g_k, \quad \gamma = \frac{1}{g} g_k l^k, \quad (3.3)$$

so that the A -special relation (2.5) takes on the form

$$A_{i|k} = \frac{1}{g} g_k A_i + \eta \mathcal{H}_{ik}. \quad (3.4)$$

We say that the space $\mathcal{F}\mathcal{F}_g^{PD}$ is *b-parallel*, if the 1-form b is parallel in the sense of the associated Riemannian space, that is, when

$$\nabla_i b_j = 0. \quad (3.5)$$

It proves that the following theorem is valid.

THEOREM 3.1. In the b -parallel involutive space $\mathcal{IF}\mathcal{F}_g^{PD}$ the A -special relation (3.4) holds.

See Appendix C, in which all the involved evaluations (which are not short) have been presented and the representation (C.19) has been arrived at, which belongs to the type (3.4). The η entered the right-hand part of (3.4) can be written down from (C.19).

As a direct consequence of the above theorem,

$$\{\nabla b = 0 \text{ and } dg = \mu b\} \implies \dot{\alpha}_i = 0. \quad (3.6)$$

From (2.4) and (3.1) we have

$$\gamma = \frac{1}{K} \frac{\mu}{g} b. \quad (3.7)$$

With this formula, the current (2.22) takes on the explicit representation

$$J_j = \frac{1}{4K} \mu g b \left(h^l_u h^m_k - h^l_k h^m_u \right) \left(-R^u_{lj} \delta^k_m + \frac{1}{2} R^u_{lm} \delta^k_j \right) \quad (3.8)$$

which is proportional to the involution scalar μ .

4. Conclusions

The Finsleroid–Finsler space involves a characteristic scalar, $g(x)$, such that the vanishing of the scalar reduces the space to a Riemannian space. Varying $g(x)$ entails varying the form of the Finsleroid. The Landsberg case of the Finsleroid–Finsler space implies strictly $g = \text{const}$, as a direct consequence of the formulas (1.13)–(1.15). To set a liberty to the scalar $g(x)$, we must overcome the restrictive case. It proves that a fruitful idea is to substitute the condition $\dot{\alpha}_{ijk} = 0$ with the Landsberg condition $\dot{A}_{ijk} = 0$ proper. Would one assume $\|A\| = \text{const}$, one observes that $\dot{\alpha}_{ijk} = 0$ implies $\dot{A}_{ijk} = 0$. In the Finsleroid–Finsler space under study, $g \neq \text{const}$ implies $\|A\| \neq \text{const}$ (see (1.17)).

The involution condition (3.1) can be written as $dg = \mu b_i(x) dx^i$ which means geometrically that the scalar $g(x)$ varies in the direction assigned by the vector $b_i(x)$.

The obtained involutive curvature tensor R^i_k (given by (C.29)) is of the novel type, being created by the gradient of the Finsleroid charge and constructed from the involutive spray coefficients E^k . The tensor is meaningful even if the associated Riemannian space is flat.

It would be appealing to develop in future the extensions which can go over the b -parallel case $\nabla b = 0$.

We have examined the conservation law for the fundamental tensor ρ_{ij} , obtaining the result (2.20)–(2.22). In the Landsberg case, the hv -curvature tensor P_{ijkl} is well-known to be totally symmetric in all four of its indices (see p. 60 in [2]), such that the skew-part $P^{[lm]}_{ku}$, and whence the right-hand part of (2.20), vanishes. Under the A -special condition, however, the tensor is meaningful, being proportional to the indicatrix curvature tensor in accordance with (2.15), so that the current J_j given by (2.21) is not the zero.

Various Finslerian ideas of applications (see [8-10]) can well be matched to the ($g \neq \text{const}$)-Finsleroid-Finsler space.

Appendix A: Evaluation of quantities of the space \mathcal{F}_g^{PD}

Below, we evaluate the key objects of the space \mathcal{F}_g^{PD} under the general setting when the Finsleroid charge g may depend on x , so that $g = g(x)$. The unit norm $\|b\| = 1$ of the input 1-form b is assumed. Our treatment will be of *local* character. Any dimension $N \geq 2$ is admissible. The Riemannian squared length $S^2 = b^2 + q^2$ underlines the Finslerian space under study.

It is appropriate to use the variables

$$u_i := a_{ij}y^j, \quad v^i := y^i - bb^i, \quad v_m := u_m - bb_m = r_{mn}y^n \equiv a_{mn}v^n, \quad (\text{A.1})$$

where $r_{mn} = a_{mn} - b_m b_n$. We obtain the relations

$$r_{ij} = \frac{\partial v_i}{\partial y^j}, \quad (\text{A.2})$$

$$u_i v^i = v_i y^i = q^2, \quad v_i b^i = v^i b_i = 0, \quad (\text{A.3})$$

$$r_{in}v^n = v_i, \quad v_kv^k = q^2, \quad (\text{A.4})$$

and

$$\frac{\partial b}{\partial y^i} = b_i, \quad \frac{\partial q}{\partial y^i} = \frac{v_i}{q}. \quad (\text{A.5})$$

In terms of the variable

$$w = \frac{q}{b}, \quad (\text{A.6})$$

we obtain

$$\frac{\partial w}{\partial y^i} = \frac{z_i}{b^2 q}, \quad z_i = bv_i - q^2 b_i \equiv bu_i - S^2 b_i, \quad (\text{A.7})$$

and

$$y^i z_i = 0, \quad b^i z_i = b^2 - S^2,$$

together with

$$a^{ij} z_i z_j = S^2 (S^2 - b^2) \equiv S^2 b^2 \lambda,$$

where

$$\lambda = w^2 \equiv \frac{1}{b^2} (S^2 - b^2). \quad (\text{A.8})$$

We also introduce the η -tensor by means of the components

$$\eta_{ij} := r_{ij} - \frac{1}{q^2} v_i v_j, \quad \eta^i_j := r^i_j - \frac{1}{q^2} v^i v_j, \quad \eta^{ij} := r^{ij} - \frac{1}{q^2} v^i v^j. \quad (\text{A.9})$$

It follows directly that

$$\eta^n_j = a^{nm} \eta_{mj}, \quad \eta^{ij} = a^{in} \eta_n^j, \quad (\text{A.10})$$

$$\eta_{mi} y^i = 0, \quad (\text{A.11})$$

$$\eta_{ij} b^j = 0, \quad \eta_{ij} z^j = 0, \quad a^{ij} \eta_{ij} = N - 2, \quad (\text{A.12})$$

and

$$\frac{\partial \left(\frac{1}{q} v_k \right)}{\partial y^j} = \frac{1}{q} \eta_{kj}, \quad \frac{\partial \eta_{ij}}{\partial y^k} = -\frac{1}{q^2} (v_i \eta_{jk} + v_j \eta_{ik}). \quad (\text{A.13})$$

We shall also use the vector

$$e_k := \frac{b}{q^2} v_k - b_k \equiv \frac{b}{q^2} u_k - \frac{S^2}{q^2} b_k, \quad (\text{A.14})$$

obtaining

$$e_k = -q \frac{\partial \left(\frac{b}{q} \right)}{\partial y^k}, \quad (\text{A.15})$$

$$\frac{\partial e_k}{\partial y^j} = \frac{b}{q^2} \eta_{kj} - \frac{1}{q^2} v_k e_j = \frac{b}{q^2} \eta_{kj} - \frac{1}{b} (e_k + b_k) e_j, \quad (\text{A.16})$$

$$e_k y^k = 0, \quad (\text{A.17})$$

$$e_k b^k = -\frac{1}{w^2} \lambda, \quad e^j \eta_{ij} = 0, \quad \frac{\partial \eta_{ij}}{\partial y^k} = -\frac{1}{b} (e_i \eta_{jk} + b_i \eta_{jk} + e_j \eta_{ik} + b_j \eta_{ik}), \quad (\text{A.18})$$

and

$$z_k = q^2 e_k, \quad (\text{A.19})$$

together with

$$w^4 a^{ij} e_i e_j = (1 + w^2) \lambda \equiv \frac{S^2}{b^2} \lambda. \quad (\text{A.20})$$

Using the generating function $V = V(x, w)$ defined from the representation

$$K = bV, \quad (\text{A.21})$$

we obtain

$$\frac{\partial K}{\partial y^i} = b_i V + \frac{1}{bq} z_i V', \quad \frac{\partial^2 K}{\partial y^i \partial y^j} = \frac{1}{q} \eta_{ij} V' + \frac{1}{b^3 q^2} z_i z_j V''. \quad (\text{A.22})$$

The prime $\{'\}$ means differentiation with respect to w . Taking into account the Finslerian rules

$$l_i = \frac{\partial K}{\partial y^i}, \quad y_i = K l_i, \quad h_{ij} = K \frac{\partial^2 K}{\partial y^i \partial y^j}, \quad g_{ij} = h_{ij} + l_i l_j, \quad (\text{A.23})$$

from (A.22) we find the representations

$$g_{ij} = \frac{1}{w} V V' \eta_{ij} + \frac{V V'}{bq} (b_i z_j + b_j z_i) + V^2 b_i b_j + \frac{1}{b^2 q^2} (V V'' + (V')^2) z_i z_j \quad (\text{A.24})$$

and

$$h_{ij} = \frac{1}{w} V V' \eta_{ij} + \frac{1}{b^2 q^2} V V'' z_i z_j. \quad (\text{A.25})$$

The determinant of the metric tensor is found to read

$$\det(g_{ij}) = \frac{1}{w^2} \gamma \left(\frac{1}{w} V V' \right)^{N-2} V^3 \det(a_{ij}) \quad (\text{A.26})$$

with

$$\gamma = w^2 V''. \quad (\text{A.27})$$

Below, the scalar $g = g(x)$ is specified as follows:

$$-2 < g(x) < 2. \quad (\text{A.28})$$

We shall apply the convenient notation

$$h = \sqrt{1 - \frac{1}{4} g^2}, \quad G = \frac{g}{h}. \quad (\text{A.29})$$

The *Finsleroid-characteristic quadratic form*

$$B(x, y) := b^2 + gbq + q^2 \equiv \frac{1}{2} \left[(b + g_+ q)^2 + (b + g_- q)^2 \right] > 0, \quad (\text{A.30})$$

where $g_+ = \frac{1}{2}g + h$ and $g_- = \frac{1}{2}g - h$, is of the negative discriminant

$$D_{\{B\}} = -4h^2 < 0 \quad (\text{A.31})$$

and, therefore, is positively definite.

We shall use also the function $\tau(x, w)$ defined by

$$B = b^2 \tau, \quad (\text{A.32})$$

obtaining from (A.30) the quadratic-case representation

$$\tau = 1 + g(x)w + w^2. \quad (\text{A.33})$$

We use this function to produce the generating function V according to the rule

$$V = \exp \int \frac{w dw}{\tau}. \quad (\text{A.34})$$

Since the function (A.33) is representable in the form

$$\tau = h^2 + \left(w + \frac{g}{2}\right)^2, \quad (\text{A.35})$$

the integration process in (A.34) is simple, namely, the resultant Finslerian metric function $K = bV$ (see (A.21)) is given by the following definition.

Definition. The scalar function $K(x, y)$ given by the formulas

$$K(x, y) = \sqrt{B(x, y)} J(x, y) \quad (\text{A.36})$$

and

$$J(x, y) = e^{-\frac{1}{2}G(x)f(x, y)}, \quad (\text{A.37})$$

where

$$f = -\arctan \frac{G}{2} + \arctan \frac{L}{hb}, \quad \text{if } b \geq 0, \quad (\text{A.38})$$

and

$$f = \pi - \arctan \frac{G}{2} + \arctan \frac{L}{hb}, \quad \text{if } b \leq 0, \quad (\text{A.39})$$

with

$$L = q + \frac{g}{2}b, \quad (\text{A.40})$$

is called the *Finsleroid-Finsler metric function*.

The function K has been normalized such that

$$0 \leq f \leq \pi,$$

$$f = 0, \quad \text{if } q = 0 \quad \text{and } b > 0; \quad f = \pi, \quad \text{if } q = 0 \quad \text{and } b < 0,$$

and the Finsleroid length $K(x, b^i(x))$ of the vector b^i is equal to the Riemannian length scalar $\|b\| = 1$, such that

$$K(x, b^i(x)) = 1. \quad (\text{A.41})$$

Sometimes it is convenient to use also the function

$$A = b + \frac{g}{2}q. \quad (\text{A.42})$$

The identities

$$L^2 + h^2 b^2 = B, \quad A^2 + h^2 q^2 = B \quad (\text{A.43})$$

are valid.

The zero-vector $y = 0$ is excluded from consideration. The positive (not absolute) homogeneity holds:

$$K(x, \lambda y) = \lambda K(x, y), \quad \lambda > 0, \quad \forall x, \quad \forall y.$$

Given the function K of the form (A.36), the generating function is obtained from (A.32) to read

$$V = \tau J \quad (\text{A.44})$$

Using (A.37), it is easy to verify that

$$(\ln V)' = \frac{w}{\tau}, \quad (\text{A.45})$$

which manifests that the integral representation (A.34) takes place.

Definition. The arisen space

$$\mathcal{FF}_g^{PD} := \{\mathcal{R}_N; b_i(x); g(x); K(x, y)\} \quad (\text{A.46})$$

is called the *Finsleroid-Finsler space*.

Definition. The space \mathcal{R}_N entering the above definition is called the *associated Riemannian space*.

Definition. Within any tangent space $T_x M$, the Finsleroid-metric function $K(x, y)$ produces the *Finsleroid*

$$\mathcal{F}_g^{PD} := \{y \in \mathcal{F}_g^{PD} : y \in T_x M, K(x, y) \leq 1\}. \quad (\text{A.47})$$

Definition. The *Finsleroid Indicatrix* $I_{g\{x\}}^{PD} \in T_x M$ is the boundary of the Finsleroid:

$$I_{g\{x\}}^{PD} := \{y \in I_{g\{x\}}^{PD} : y \in T_x M, K(x, y) = 1\}. \quad (\text{A.48})$$

Since at $g = 0$ the \mathcal{FF}_g^{PD} -space is Riemannian, then the body $\mathcal{F}_{g=0\{x\}}^{PD}$ is a unit ball and $I_{g=0\{x\}}^{PD}$ is a unit sphere.

Definition. The scalar $g(x)$ is called the *Finsleroid charge*. The 1-form $b = b_i(x)y^i$ is called the *Finsleroid-axis 1-form*.

The determinant (A.26) takes on the form

$$\det(g_{ij}) = V^{2N} \frac{1}{\tau^N} \det(a_{ij}). \quad (\text{A.49})$$

The contravariant components g^{ij} of the associated Finslerian metric tensor can be given by the representation

$$\frac{1}{w} V V' g^{ij} = a^{ij} + p b^i b^j + r (b^i y^j + b^j y^i) + t y^i y^j \quad (\text{A.50})$$

with

$$r = -\frac{g}{bw}, \quad p + br = 0, \quad t = \frac{g}{B}(1 + gw). \quad (\text{A.51})$$

Therefore,

$$\frac{K^2}{B}g^{ij} = a^{ij} + \frac{g}{w}b^ib^j - \frac{g}{bw}(b^iy^j + b^jy^i) + \frac{g}{Bw}(1 + gw)y^iy^j. \quad (\text{A.52})$$

From (A.24) it follows that

$$\frac{B}{K^2}g_{ij} = \eta_{ij} + w^2(b_ie_j + b_je_i) + \tau b_ib_j + \frac{w^2}{\tau}(\tau - gw)e_ie_j. \quad (\text{A.53})$$

In this way we obtain

$$\frac{B}{K^2}g_{ij} = a_{ij} + \frac{g}{B}\left[Bwb_ib_j - \frac{1}{w}v_iv_j + bw(b_iv_j + b_jv_i) - b^2w^3b_ib_j\right].$$

Eventually, we obtain the Finsleroid metric tensor representation

$$\frac{B}{K^2}g_{ij} = a_{ij} + \frac{g}{B}\left[b^2(1 + gw)wb_ib_j + bw(b_iv_j + b_jv_i) - \frac{1}{w}v_iv_j\right]. \quad (\text{A.54})$$

We can explicate the associated vector

$$A_k = g^{ij}A_{ijk} \quad (\text{A.55})$$

by applying the known general formula

$$A_k = K \frac{\partial \ln \left(\sqrt{\det(g_{ij})} \right)}{\partial y^k},$$

so that

$$A_k = \left(\ln \left(\sqrt{\det(g_{ij})} \right) \right)' K \frac{\partial w}{\partial y^k}. \quad (\text{A.56})$$

From (A.52) it follows that

$$\frac{1}{w}VV'g^{kn}b_k = b^n - g\frac{w}{b\tau}y^n. \quad (\text{A.57})$$

The first member of (A.22) entails the equality

$$y_k = \frac{1}{b}K^2 \left(b_k + \frac{b^2w^2}{B}e_k \right). \quad (\text{A.58})$$

With

$$K \frac{\partial w}{\partial y^k} = Vwe_k$$

(see (A.7) and (A.19)) and (A.49), the representation (A.56) of the vector A_k is found to read

$$A_k = -\frac{KNg}{2B}qe_k. \quad (\text{A.59})$$

Using here (A.58), we may also write

$$A_k = \frac{NK}{2}g\frac{1}{bw} \left(b_k - \frac{b}{K^2}y_k \right). \quad (\text{A.60})$$

Another convenient form is

$$A_k = \frac{NK}{2}g\frac{1}{qB}(q^2b_k - bv_k). \quad (\text{A.61})$$

From (A.57) and (A.60) it follows immediately that

$$A^k = \frac{Ng}{2Kbw} \left[Bb^k - b(1 + gw)y^k \right]. \quad (\text{A.62})$$

It can readily be seen that the representations (A.60) and (A.62) entail

$$A^k A_k = \frac{N^2 g^2}{4}. \quad (\text{A.63})$$

We have

$$y_k b^k = b \frac{K^2}{B} (1 + gw), \quad (\text{A.64})$$

$$A_k b^k = \frac{KNg}{2B} bw, \quad b_k A^k = \frac{Ng}{2K} bw, \quad (\text{A.65})$$

and

$$Kg^{kj}b_j = \frac{2bw}{Ng}A^k + bl^k, \quad (\text{A.66})$$

together with

$$K^2 g^{kj}b_j = Bb^k - gbwy^k. \quad (\text{A.67})$$

The relation (A.59) can be inverted, yielding

$$e_k = -\frac{2B}{KNgq}A_k. \quad (\text{A.68})$$

Taking into account the formulas (A.7), (A.19), and (A.68), we may write simply

$$\frac{\partial w}{\partial y^i} = -\frac{2B}{b^2} \frac{1}{KNg} A_i. \quad (\text{A.69})$$

This formula is convenient to use in many involved evaluations.

With (A.52), it follows that

$$Kg^{kj}g_j = \frac{1}{K}B(g^k - (bg)b^k) + \frac{2bw}{Ng}(bg)A^k + \frac{2}{N}[b(bg) - (yg)]A^k + b(bg)l^k. \quad (\text{A.70})$$

Now we perform differentiation of the functions indicated in (A.29) with respect to the Finsleroid parameter g , obtaining

$$\frac{\partial h}{\partial g} = -\frac{1}{4}G, \quad \frac{\partial G}{\partial g} = \frac{1}{h^3}, \quad \frac{\partial \left(\frac{G}{h} \right)}{\partial g} = \frac{1}{h^4} \left(1 + \frac{g^2}{4} \right), \quad (\text{A.71})$$

$$\frac{\partial f}{\partial g} = -\frac{1}{2h} + \frac{b}{B} \left(\frac{1}{4}Gq + \frac{1}{2h}b \right), \quad (\text{A.72})$$

and

$$2K \frac{\partial K}{\partial g} = bqJ^2 - \frac{1}{h^3}fK^2 + G \left[\frac{1}{2h} - \frac{b}{B} \left(\frac{1}{4}Gq + \frac{1}{2h}b \right) \right] K^2, \quad (\text{A.73})$$

or

$$\frac{\partial K^2}{\partial g} = MK^2, \quad (\text{A.74})$$

where

$$M = \frac{bq}{B} - \frac{1}{h^3}f + \frac{1}{2} \frac{G}{hB} (q^2 + \frac{1}{2}gbq),$$

or

$$M = -\frac{1}{h^3}f + \frac{1}{2} \frac{G}{hB} q^2 + \frac{1}{h^2B} bq. \quad (\text{A.75})$$

Using

$$e_i = -b_i + \frac{b}{q^2}v_i,$$

we find

$$M_i = \frac{4q^2}{gNBK} A_i. \quad (\text{A.76})$$

Differentiating (A.74) with respect to y^i just yields

$$\frac{\partial y_i}{\partial g} = My_i + \frac{1}{2}M_i K^2. \quad (\text{A.77})$$

In view of (A.59) and (A.76), we may write

$$\frac{\partial y_i}{\partial g} = My_i - \frac{q^2 q K^2}{B^2} e_i, \quad (\text{A.78})$$

or

$$\frac{\partial y_i}{\partial g} = My_i + \frac{2q^2 K}{gNB} A_i. \quad (\text{A.79})$$

With the tensor

$$\mathcal{H}_{ij} = h_{ij} - \frac{A_i A_j}{A_n A^n}, \quad (\text{A.80})$$

we can arrive at

$$\frac{\partial g_{ij}}{\partial g} = Mg_{ij} + \frac{q^2}{B} \frac{2}{gN} (A_i l_j + l_i A_j) - \frac{bq}{B} \mathcal{H}_{ij} - \frac{2bq}{B} \frac{4}{g^2 N^2} A_i A_j. \quad (\text{A.81})$$

From (A.61) we obtain the simple result

$$\frac{\partial A_i}{\partial g} = \left(\frac{1}{2}M + \frac{1}{g} - \frac{b}{B}q \right) A_i. \quad (\text{A.82})$$

From (A.62) it follows that

$$\frac{\partial \left(\frac{K}{Ng} A^k \right)}{\partial g} = -\frac{1}{2} (y^k - bb^k).$$

From (A.75) we get

$$\frac{\partial M}{\partial g} = \frac{3g}{4h^2}M + \frac{1}{h^2}\frac{q^2}{B} - \frac{1}{h^2}\frac{q^2}{B^2}\left(b^2 + \frac{1}{2}gbq\right).$$

Next, with (A.62) we now evaluate the derivative

$$\frac{\partial A^k}{\partial y^n} = -\frac{1}{K}A_n l^k - \frac{Ng}{2Kw}(1+gw)\left(\delta^k_n - l^k l_n - \frac{4}{N^2 g^2}A^k A_n\right) - \frac{2}{N}\frac{1}{K}A^k A_n. \quad (\text{A.83})$$

The previous representation can be written as

$$\frac{\partial A^k}{\partial y^n} = -\frac{1}{K}A_n l^k - \frac{Ng}{2Kw}(1+gw)\mathcal{H}^k_n - \frac{2}{N}\frac{1}{K}A^k A_n \quad (\text{A.84})$$

in terms of the tensor

$$\mathcal{H}^k_n = \delta^k_n - l^k l_n - \frac{4}{N^2 g^2}A^k A_n \quad (\text{A.85})$$

given by (A.80).

The Cartan tensor

$$A_{ijk} = \frac{1}{2}K\frac{\partial g_{ij}}{\partial y^k} \quad (\text{A.86})$$

takes on the form

$$A_{ijk} = \frac{1}{N}\left(h_{jk}A_i + h_{ik}A_j + h_{ij}A_k - \frac{1}{A^h A_h}A_i A_j A_k\right), \quad (\text{A.87})$$

or in terms of the tensor (A.85),

$$A_{ijk} = \frac{1}{N}\left(\mathcal{H}_{jk}A_i + \mathcal{H}_{ik}A_j + \mathcal{H}_{ij}A_k + \frac{2}{A^h A_h}A_i A_j A_k\right). \quad (\text{A.88})$$

We can conclude that

$$A^k A_{ijk} = \frac{1}{N}(A_i A_j + h_{ij}A_k A^k) \equiv \frac{1}{N}(2A_i A_j + \mathcal{H}_{ij}A_k A^k). \quad (\text{A.89})$$

Also, we find

$$A_{ijk}A^k_{mn} = \frac{1}{N}A_i A_{jmn} + \frac{1}{N}A_j A_{imn} + \frac{2}{N^2}\mathcal{H}_{ij}A_m A_n + \frac{1}{N}\frac{1}{N}A_k A^k \mathcal{H}_{ij}\mathcal{H}_{mn}. \quad (\text{A.90})$$

From (A.87) it follows that

$$A^{ijk}A_{ijk} = \frac{1}{N}\left(3 - \frac{2}{N}\right)A^h A_h. \quad (\text{A.91})$$

The curvature of indicatrix is well-known to be described by the tensor

$$\hat{R}^j_{i\ mn} := \frac{1}{K^2}\left(A_h^j{}_m A_i^h{}_n - A_h^j{}_n A_i^h{}_m\right). \quad (\text{A.92})$$

Inserting (A.90) in (A.92), we find that

$$K^2 \hat{R}_{ijmn} = \frac{1}{N^2}(A^k A_k)\left(h_{in}h_{mj} - h_{im}h_{nj}\right). \quad (\text{A.93})$$

Contracted objects

$$R_{im} = \hat{R}_i^j{}_{mj} \quad (\text{A.94})$$

and

$$R = \hat{R}_i^j{}_{mj} g^{im} \quad (\text{A.95})$$

are found to be

$$R_{im} = \frac{1}{K^2} \left(\frac{2}{N} - 1 \right) \frac{1}{N} (A^k A_k) h_{im} \quad (\text{A.96})$$

and

$$R = \frac{1}{K^2} \left(\frac{2}{N} - 1 \right) (N - 1) \frac{1}{N} (A^k A_k). \quad (\text{A.97})$$

If we consider the derivative

$$\frac{\partial A_k}{\partial y^n} = g_{km} \frac{\partial A^m}{\partial y^n} + \frac{2}{K} A_{kmn} A^m$$

and apply (A.84) together with (A.89), we obtain

$$\frac{\partial A_k}{\partial y^n} = -\frac{1}{K} A_n l_k - \frac{Ng}{2Kw} \mathcal{H}_{kn} + \frac{2}{N} \frac{1}{K} A_k A_n. \quad (\text{A.98})$$

Also,

$$\tau_{ij} := A_{ij} - A_k A_i^k{}_j, \quad (\text{A.99})$$

where

$$A_{ij} := K \frac{\partial A_i}{\partial y^j} + l_i A_j. \quad (\text{A.100})$$

The identities

$$\tau_{ij} y^j = 0, \quad \tau_{ij} A^j = 0$$

hold. We obtain

$$\tau_{ij} = -\frac{N}{4} \frac{g(2b + gq)}{q} \mathcal{H}_{ij}. \quad (\text{A.101})$$

Differentiating (A.85) leads to

$$K \frac{\partial \mathcal{H}^k{}_n}{\partial y^m} = -\mathcal{H}^k{}_m l_n - l^k \mathcal{H}_{nm} + \frac{2}{Ng} \frac{1}{w} \mathcal{H}_{nm} A^k + \frac{2}{Ng} \frac{1}{w} (1 + gw) \mathcal{H}^k{}_m A_n. \quad (\text{A.102})$$

If we apply (A.98) to (A.87), we get

$$\begin{aligned} \frac{\partial A_{ijk}}{\partial y^n} &= \frac{1}{K} \frac{2}{N} (A_{jkn} A_i + A_{ikn} A_j + A_{ijn} A_k) - \frac{1}{K} (l_j A_{kni} + l_i A_{knj} + l_k A_{ijn}) \\ &\quad + \frac{1}{K} \frac{1}{N} \frac{2}{N} [\mathcal{H}_{jk} A_i A_n + \mathcal{H}_{ik} A_j A_n + \mathcal{H}_{ij} A_k A_n] \\ &\quad - \frac{g}{2Kw} [\mathcal{H}_{jk} \mathcal{H}_{in} + \mathcal{H}_{ik} \mathcal{H}_{jn} + \mathcal{H}_{ji} \mathcal{H}_{kn}]. \end{aligned} \quad (\text{A.103})$$

If we introduce the tensor

$$\tau_{ijkn} := K \frac{\partial A_{ijk}}{\partial y^n} + l_j A_{kni} + l_i A_{knj} + l_k A_{ijn} - A_{jkm} A^m_{in} - A_{ikm} A^m_{jn} - A_{jim} A^m_{kn}, \quad (\text{A.104})$$

then we can conclude from (A.103) and (A.90) that

$$\tau_{ijkn} = -\frac{g(2+gw)}{4w} \left[\mathcal{H}_{jk} \mathcal{H}_{in} + \mathcal{H}_{ik} \mathcal{H}_{jn} + \mathcal{H}_{ji} \mathcal{H}_{kn} \right]. \quad (\text{A.105})$$

In calculating the Finsleroid-case spray coefficients there appears the vector

$$E^k := M(yg)y^k + \frac{1}{2}K^2(yg)M_h g^{kh} - \frac{1}{2}MK^2 g_h g^{kh}. \quad (\text{A.106})$$

Noting (A.76), we obtain

$$E^k = M(yg)y^k + K \frac{2b^2 w^2}{gNB} (yg) A^k - \frac{1}{2}MK^2 g_h g^{kh}. \quad (\text{A.107})$$

We may calculate the derivative

$$E^k_n := \frac{\partial E^k}{\partial y^n}. \quad (\text{A.108})$$

We obtain

$$\begin{aligned} E^k_n &= MT^k_n + K \frac{2b^2 w^2}{gNB} A^k g_n \\ &+ \frac{2b^2 w^2}{gNB} (yg) A^k l_n - \frac{4w}{gN} (yg) \frac{2}{Ng} A_n A^k \\ &+ \frac{4b^2 w^3}{gNB} (yg) \frac{2}{Ng} A_n A^k + \frac{2b^2 w^2}{gNB} (yg) \left[A_n l^k - \frac{Ng}{2w} (1+gw) \mathcal{H}^k_n \right] - \frac{1}{2}K^2 M_n g_h g^{kh} \end{aligned} \quad (\text{A.109})$$

with

$$T^k_n = g_n y^k + (yg) \delta^k_n - \frac{1}{2}K g_h g^{kh} l_n - \frac{1}{2}K \frac{\partial (K g_h g^{kh})}{\partial y^n}.$$

The contraction $K g_h g^{kh}$ is to be taken from (A.70).

Using (A.81) yields

$$\begin{aligned} &\left(\frac{\partial g_{kj}}{\partial g} g_i + \frac{\partial g_{ik}}{\partial g} g_j - \frac{\partial g_{ij}}{\partial g} g_k \right) A^k = \\ &\left(MA_j + \frac{q^2}{B} \frac{gN}{2} l_j - \frac{2bq}{B} A_j \right) g_i + \left(MA_i + \frac{q^2}{B} \frac{gN}{2} l_i - \frac{2bq}{B} A_i \right) g_j \\ &- \left(Mg_{ij} + \frac{q^2}{B} \frac{2}{gN} (A_i l_j + l_i A_j) - \frac{bq}{B} \mathcal{H}_{ij} - \frac{2bq}{B} \frac{4}{g^2 N^2} A_i A_j \right) g_k A^k. \end{aligned} \quad (\text{A.110})$$

We can write down simple explicit representations for the partial derivatives with respect to x , using the notation

$$g_j = \frac{\partial g}{\partial x^j}, \quad s_k = y^m \nabla_k b_m. \quad (\text{A.111})$$

We may use the equality

$$\frac{\partial w}{\partial x^k} = -\frac{1}{b^2 q} S^2 s_k + \Delta \quad (\text{A.112})$$

(see (A.6)), where Δ symbolizes the summary of the terms which involve partial derivatives of the input Riemannian metric tensor a_{ij} with respect to the coordinate variables x^k . We use the Riemannian covariant derivative

$$\nabla_i b_j := \partial_i b_j - b_k a^k_{ij}, \quad (\text{A.113})$$

where

$$a^k_{ij} := \frac{1}{2} a^{kn} (\partial_j a_{ni} + \partial_i a_{nj} - \partial_n a_{ji}) \quad (\text{A.114})$$

are the Christoffel symbols given rise to by the associated Riemannian metric.

First of all, we differentiate the quadratic form B given by (A.30), obtaining

$$\frac{\partial B}{\partial x^j} = b q g_j + \frac{2}{b} (B - S^2) s_j - g \frac{1}{q} S^2 s_j + \Delta. \quad (\text{A.115})$$

Also, starting with (A.21), we find

$$\frac{\partial K}{\partial x^j} = \frac{1}{2} M K g_j + K \frac{b}{B} g w s_j + \Delta, \quad (\text{A.116})$$

where we have used (A.74).

Eventually, the following sufficiently simple representation is obtained:

$$\begin{aligned} \frac{\partial A_i}{\partial x^j} &= \left(\frac{1}{2} M + \frac{1}{g} - \frac{b}{B} q \right) A_i g_j + \frac{N K}{2} g \frac{1}{q B} S^2 \nabla_j b_i \\ &+ \frac{b}{B} g w s_j A_i - \frac{1}{B} \left(\frac{2}{b} (B - S^2) s_j - g \frac{1}{q} S^2 s_j \right) A_i \\ &+ \frac{1}{w^2} \frac{1}{b^3} S^2 s_j A_i - \frac{N K}{2} g \frac{1}{q B} \frac{1}{b} S^2 s_j b_i + \Delta. \end{aligned} \quad (\text{A.117})$$

Appendix B: Finsleroid–Finsler spray coefficients

Evaluations involve the induced *spray coefficients*

$$G^k = \gamma^k_{ij} y^i y^j, \quad (\text{B.1})$$

which entail the coefficients

$$G^i_k : = \frac{\partial G^i}{\partial y^k}, \quad G^i_{km} : = \frac{\partial G^i_k}{\partial y^m}, \quad G^i_{kmn} : = \frac{\partial G^i_{km}}{\partial y^n}, \quad (\text{B.2})$$

and

$$\bar{G}^i = \frac{1}{2}G^i, \quad \bar{G}^i{}_k = \frac{1}{2}G^i{}_k, \quad \bar{G}^i{}_{km} = \frac{1}{2}G^i{}_{km}, \quad \bar{G}^i{}_{kmn} = \frac{1}{2}G^i{}_{kmn}. \quad (\text{B.3})$$

The homogeneity gives rise to the identities

$$2G^i = G^i{}_k y^k, \quad G^i{}_k = G^i{}_{km} y^m, \quad G^i{}_{kmn} y^n = 0. \quad (\text{B.4})$$

The pair (x, y) , — the so-called *line element*, — is the argument of the Finslerian objects;

$$\gamma^k{}_{ij} := \frac{1}{2}g^{kn} \left(\frac{\partial g_{ni}}{\partial x^j} + \frac{\partial g_{nj}}{\partial x^i} - \frac{\partial g_{ji}}{\partial x^n} \right) \quad (\text{B.5})$$

are the Christoffel symbols given rise to by the Finsleroid–Finsler metric function K .

Below, the abbreviation h means *horizontal*. On the basis of the above coefficients the Finslerian *connection coefficients* $\Gamma^k{}_{ij}$ of h -type are constructed according to the well-known rule:

$$\Gamma^k{}_{ij} = \gamma^k{}_{ij} - \bar{G}^n{}_i C_n{}^k{}_j - \bar{G}^n{}_j C_n{}^k{}_i + \bar{G}^{kn} C_{nij} \quad (\text{B.6})$$

with

$$\bar{G}^n{}_i = \gamma^n{}_{ij} y^j - 2\bar{G}^m C_m{}^n{}_i = \Gamma^n{}_{ij} y^j = \frac{1}{2}G^n{}_i \quad (\text{B.7})$$

and

$$2\bar{G}^m = \gamma^m{}_{ij} y^i y^j = \bar{G}^m{}_i y^i = \Gamma^m{}_{ij} y^i y^j = G^m, \quad (\text{B.8})$$

where $C_{nij} = A_{nij} K^{-1}$ and $C_n{}^k{}_j = A_n{}^k{}_j K^{-1}$. By the help of these coefficients the *h-covariant derivatives* of tensors are constructed as exemplified by

$$A_{i|j} := \partial_j A_i - \bar{G}^k{}_j \frac{\partial A_i}{\partial y^k} - \Gamma^k{}_{ij} A_k \quad (\text{B.9})$$

(see [1,2]).

For the hh -curvature tensor $R^i{}_k$ we use the formula

$$K^2 R^i{}_k := 2 \frac{\partial \bar{G}^i}{\partial x^k} - \frac{\partial \bar{G}^i}{\partial y^j} \frac{\partial \bar{G}^j}{\partial y^k} - y^j \frac{\partial^2 \bar{G}^i}{\partial x^j \partial y^k} + 2\bar{G}^j \frac{\partial^2 \bar{G}^i}{\partial y^k \partial y^j} \quad (\text{B.10})$$

(which is tantamount to the definition (3.8.7) on p. 66 of the book [2]). The concomitant tensors

$$R^i{}_{km} := \frac{1}{3K} \left(\frac{\partial(K^2 R^i{}_k)}{\partial y^m} - \frac{\partial(K^2 R^i{}_m)}{\partial y^k} \right), \quad (\text{B.11})$$

and

$$R_n{}^i{}_{km} := \frac{\partial(K R^i{}_{km})}{\partial y^n} - \left(\dot{A}^i{}_{nm|k} - \dot{A}^i{}_{nk|m} + \dot{A}^i{}_{uk} \dot{A}^u{}_{nm} - \dot{A}^i{}_{um} \dot{A}^u{}_{nk} \right) \quad (\text{B.12})$$

(see p. 60 in the book [2]) arise. We have

$$R^i{}_k y^k = 0 \quad (\text{B.13})$$

and

$$R^i{}_{km} y^m = K R^i{}_k. \quad (\text{B.14})$$

In calculations, it proves convenient to write the derivative (B.9) in the alternative form

$$A_{i|j} = \partial_j A_i - \bar{G}^k{}_j \tau_{ik} \frac{1}{K} - \tilde{\Gamma}^k{}_{ij} A_k + \bar{G}^k{}_j A_k l_i \frac{1}{K} \quad (\text{B.15})$$

($l_i = g_{ik} l^k$) with

$$\tilde{\Gamma}^k{}_{ij} = \gamma^k{}_{ij} - \bar{G}^m{}_i A_n{}^k{}_j \frac{1}{K} + \bar{G}^{kn} A_{nij} \frac{1}{K} \quad (\text{B.16})$$

and

$$\tau_{ij} = A_{ij} - A_k A_i{}^k{}_j, \quad (\text{B.17})$$

where

$$A_{ij} = K \frac{\partial A_i}{\partial y^j} + l_i A_j. \quad (\text{B.18})$$

We know that

$$\tau_{ij} = -\frac{N}{4} \frac{g(2b + gq)}{q} \mathcal{H}_{ij} \quad (\text{B.19})$$

(see (A.101)).

By means of attentive (lengthy) evaluations we can arrive at the following assertion.

THEOREM B1. *The explicit form of the spray coefficients of the Finsleroid–Finsler space reads*

$$G^k = gq \left[a^{kj} + (pb^k + ry^k)b^j \right] y^h (\nabla_h b_j - \nabla_j b_h) + \frac{g}{q} (y^k - bb^k) (ys) + a^k{}_{mn} y^m y^n + E^k, \quad (\text{B.20})$$

where p and r are the quantities presented in (A.51) and E^k is the vector (A.107). The notation

$$(ys) = y^h s_h \quad (\text{B.21})$$

has been used, where

$$s_k = y^m \nabla_k b_m. \quad (\text{B.22})$$

A careful consideration of the formulas (B.20)–(B.22) shows that the following theorem is valid.

THEOREM B2. *The Finsleroid–Finsler space \mathcal{F}_g^{PD} is of the Landsberg type if and only if the following three conditions hold: the Finsleroid charge is a constant*

$$g = \text{const}, \quad (\text{B.23})$$

the input 1-form b is closed

$$\partial_i b_j - \partial_j b_i = 0, \quad (\text{B.24})$$

and the expansion

$$\nabla_m b_n = k (a_{mn} - b_m b_n) \quad (\text{B.25})$$

takes place, where $k = k(x)$ is a scalar.

Under the conditions of this theorem, we have $E^k = 0$ and $(ys) = kq^2$, so that the representation (B.20) reduces to

$$G^k = gkq (y^k - bb^k) + a^k{}_{mn} y^m y^n. \quad (\text{B.26})$$

It can readily be seen that at any dimension $N \geq 3$ the Berwald case corresponds to $k = 0$, so that

$$G_{\{\text{Berwaldian}\}}^m = a^m_{jn} y^j y^n. \quad (\text{B.27})$$

At the dimension $N = 2$, the Finsleroid–Finsler space \mathcal{F}_g^{PD} is of the Landsberg type if and only the space is of the Berwald type (independently of value of k).

NOTE. Theorem B2 is known from the previous publications [4-7]. Theorem B1 is a new result. When $g = \text{const}$, the coefficients E^k vanish identically, in which case the above spray coefficients (B.20) coincide with the spray coefficients given by Eq. (4.5) in [7].

Appendix C: Finsleroid–involutive tensors

Henceforth, we assume the involutive case

$$g_i = \mu b_i, \quad \mu = \mu(x), \quad (\text{C.1})$$

which entails

$$b(bg) = (yg), \quad (bg) = \mu, \quad (yg) = \mu b. \quad (\text{C.2})$$

The notation $(yg) = y^i g_i$ and $(bg) = b^i g_i$ is used; $g_i = \partial g / \partial x^i$. Under these conditions, the formula (A.70) reads merely

$$K g^{kj} g_j = \frac{2bw}{Ng} (bg) A^k + b(bg) l^k \quad (\text{C.3})$$

and the representation (A.107) reduces to read

$$E^k = \frac{1}{2} M (yg) y^k - \widehat{M} K \frac{1}{Ng} w (yg) A^k, \quad (\text{C.4})$$

where

$$\widehat{M} = M - \frac{2b^2 w}{B}. \quad (\text{C.5})$$

Differentiating this scalar yields the simple result:

$$\frac{\partial \widehat{M}}{\partial y^m} = \frac{4b^2}{B} \frac{1}{K Ng} A_m. \quad (\text{C.6})$$

The eventual representation reads

$$E^k_n = M T^k_n + \frac{4b^2 w^2}{gNB} (yg) A^k l_n - \frac{4wb^2(1+gw)}{gNB} \frac{2}{Ng} (yg) A_n A^k - \frac{b^2 w}{B} (yg) (1+gw) \mathcal{H}^k_n \quad (\text{C.7})$$

with

$$\begin{aligned} T^k_n &= (yg) l^k l_n + \frac{2}{Ng} w (yg) l^k A_n + \frac{1}{2} (yg) \mathcal{H}^k_n - \frac{2}{Ng} w (yg) A^k l_n \\ &+ \frac{2}{Ng} \frac{2(1+gw)}{Ng} (yg) A_n A^k + \frac{1}{2} (yg) (1+gw) \mathcal{H}^k_n. \end{aligned} \quad (\text{C.8})$$

We find

$$A_k E^k = -\frac{Ng}{4} K w(yg) \left(M - \frac{2b^2 w}{B} \right) \quad (\text{C.9})$$

and

$$A_k E^k{}_n = M \left(-\frac{Ng}{2} w l_n + (1 + gw) A_n \right) (yg) + \frac{2b^2 w^2}{B} \frac{Ng}{2} (yg) l_n - \frac{2wb^2(1 + gw)}{B} (yg) A_n,$$

or

$$A_k E^k{}_n = \left(M - \frac{2b^2 w}{B} \right) \left(-\frac{Ng}{2} w l_n + (1 + gw) A_n \right) (yg). \quad (\text{C.10})$$

Also,

$$E^k{}_n A^n = \left(M - \frac{2b^2 w}{B} \right) (1 + gw) (yg) A^k + M \frac{Ng}{2} w (yg) l^k \quad (\text{C.11})$$

and

$$\mathcal{H}_{km} E^k{}_n = M \left[\frac{1}{2} (yg) \mathcal{H}_{mn} + \frac{1}{2} (yg) (1 + gw) \mathcal{H}_{mn} \right] - \frac{b^2 w}{B} (yg) (1 + gw) \mathcal{H}_{mn}, \quad (\text{C.12})$$

together with

$$E^k{}_n \mathcal{H}^n{}_i = M \left[\frac{1}{2} (yg) \mathcal{H}^k{}_i + \frac{1}{2} (yg) (1 + gw) \mathcal{H}^k{}_i \right] - \frac{b^2 w}{B} (yg) (1 + gw) \mathcal{H}^k{}_i. \quad (\text{C.13})$$

We obtain

$$\begin{aligned} -E^n{}_i A_n{}^k{}_j + E^{kn} A_{nij} &= -\frac{1}{N} \left[M \frac{1}{2} (yg) \mathcal{H}_{ij} - \widehat{M} \frac{1}{2} (yg) (1 + gw) \mathcal{H}_{ij} \right] A^k \\ &\quad + \widehat{M} \frac{g}{2} w (yg) \mathcal{H}^k{}_j l_i + \widehat{M} g w (yg) \frac{1}{A^h A_h} A^k A_j l_i \\ &\quad + \frac{1}{N} \left[M \frac{1}{2} (yg) \mathcal{H}^k{}_j - \widehat{M} \frac{1}{2} (yg) (1 + gw) \mathcal{H}^k{}_j \right] A_i \\ &\quad + M \frac{g}{2} w (yg) \mathcal{H}_{ij} l^k + M g w (yg) \frac{1}{A^h A_h} A_i A_j l^k. \end{aligned} \quad (\text{C.14})$$

The last formula just entails

$$\left(-E^n{}_i A_n{}^k{}_j + E^{kn} A_{nij} \right) A_k = -\frac{Ng^2}{8} \left[M (yg) \mathcal{H}_{ij} - \widehat{M} (yg) (1 + gw) \mathcal{H}_{ij} \right] + \widehat{M} g w (yg) A_j l_i. \quad (\text{C.15})$$

The formula (A.110) can be written as

$$\left(\frac{\partial g_{kj}}{\partial g} g_i + \frac{\partial g_{ik}}{\partial g} g_j - \frac{\partial g_{ij}}{\partial g} g_k \right) A^k =$$

$$\mu\widehat{M}\left(\frac{b}{K}(A_j l_i + A_i l_j) - \frac{Ng}{2K}q l_i l_j - \frac{Ng}{2K}q \mathcal{H}_{ij} + \frac{2}{NKg}bw A_i A_j\right). \quad (\text{C.16})$$

Let us assume that the b -parallel case

$$\nabla_i b_j = 0$$

takes place. Then it proves convenient to write the derivative (B.15) in the form

$$A_{i|j} = \partial_j A_i - \frac{1}{2}E^k{}_j \tau_{ik} \frac{1}{K} - \tilde{\Gamma}^k{}_{ij} A_k + \frac{1}{2}E^k{}_j A_k l_i \frac{1}{K} + \Delta \quad (\text{C.17})$$

with

$$\tilde{\Gamma}^k{}_{ij} = \gamma^k{}_{ij} - \frac{1}{2}E^n{}_i A_n{}^k{}_j \frac{1}{K} + \frac{1}{2}E^{kn} A_{nij} \frac{1}{K} + \Delta, \quad (\text{C.18})$$

and required insertions lead to the following simple result:

$$K A_{i|j} = \frac{1}{g}K A_i g_j + \frac{1}{4}\mu\widehat{M}\frac{Ng}{2}\frac{B}{q}\mathcal{H}_{ij} + \frac{1}{4}\mu M\frac{Ng}{2}\left(\frac{b(2b+gq)}{2q} + q + gb\right)\mathcal{H}_{ij}. \quad (\text{C.19})$$

Next, we consider the derivative tensor

$$E^k{}_{nm} := \frac{\partial E^k{}_n}{\partial y^m} \quad (\text{C.20})$$

to find the contraction

$$A^n E^k{}_{nm} = \frac{\partial A^n E^k{}_n}{\partial y^m} - E^k{}_n \frac{\partial A^n}{\partial y^m}.$$

Make required cancellation and use (C.4), obtaining

$$\begin{aligned} A^n E^k{}_{nm} &= \widehat{M}[w - g(1+gw)]\frac{2}{NKg}(yg)A_m A^k \\ &\quad + \left[\widehat{M}(1+gw)A^k + M\frac{Ng}{2}wl^k\right]\frac{1}{K}(yg)l_m \\ &\quad + 2\frac{2}{KNg}(yg)A^k A_m - \widehat{M}(1+gw)(yg)\frac{1}{K}A_m l^k \\ &\quad + \frac{2q^2}{BK}w(yg)l^k A_m + M\frac{Ng}{2}(yg)w\frac{1}{K}\mathcal{H}^k{}_m + M\frac{Ng}{4Kw}(1+gw)(yg)\mathcal{H}^k{}_m. \end{aligned}$$

We get

$$E^n E^k{}_{nm} = \frac{1}{2}M(yg)E^k{}_m - \widehat{M}K\frac{1}{Ng}w(yg)A^n E^k{}_{nm}.$$

Now, write (C.7)–(C.8) as follows:

$$E^k{}_n = M\left((yg)l^k l_n + \frac{2}{Ng}w(yg)l^k A_n + \frac{1}{2}(yg)\mathcal{H}^k{}_n\right)$$

$$+ \widehat{M} \left[-\frac{2}{Ng} w(yg) A^k l_n + \frac{2}{Ng} \frac{2(1+gw)}{Ng} (yg) A_n A^k + \frac{1}{2} (yg) (1+gw) \mathcal{H}_n^k \right], \quad (\text{C.21})$$

or

$$\begin{aligned} -E^k{}_n E^n{}_m + 2E^n E^k{}_{nm} &= \frac{1}{4} M^2 (yg)^2 \mathcal{H}_m^k \\ &+ M \widehat{M} (yg)^2 \left(\frac{2}{Ng} \frac{2}{Ng} \frac{B}{b^2} A^k A_m - \frac{1}{4} (1+gw) \mathcal{H}_m^k \right) \\ &- \widehat{M}^2 (yg)^2 \left(\frac{4}{N^2 g^2} (1+gw)^2 A^k A_m + \frac{1}{4} \frac{B}{b^2} \mathcal{H}_m^k \right) \\ &- \widehat{M} \frac{2}{Ng} w(yg)^2 \left[2 \frac{2}{Ng} A^k A_m - \widehat{M} (1+gw) l^k A_m + \frac{2\tilde{q}^2}{B} w l^k A_m + M \frac{Ng}{2} w \mathcal{H}_m^k \right]. \end{aligned} \quad (\text{C.22})$$

Below we again assume that the b -parallel condition

$$\nabla_i b_j = 0$$

holds. Since

$$\frac{\partial M}{\partial x^m} = \frac{\partial M}{\partial g} g_m + \Delta,$$

we can straightforwardly come from (C.4) to

$$\frac{\partial E^k}{\partial x^m} = \frac{1}{\mu} E^k \mu_m + \frac{1}{2} (yg) y^k \frac{\partial M}{\partial g} g_m - \frac{K}{Ng} w A^k (yg) \frac{\partial \widehat{M}}{\partial g} g_m + \frac{1}{2} \widehat{M} w (yg) (y^k - b b^k) g_m + \Delta,$$

where $\mu_m = \partial \mu / \partial x^m$ and the formula placed below (A.82) has been applied. In this way we obtain

$$y^m \frac{\partial E^k}{\partial x^m} = \frac{1}{\mu} (y\mu) E^k + \frac{1}{2} (yg)^2 y^k \frac{\partial M}{\partial g} - \frac{K}{Ng} w A^k (yg)^2 \frac{\partial \widehat{M}}{\partial g} + \frac{1}{2} \widehat{M} w (yg)^2 (y^k - b b^k) + \Delta$$

and

$$2 \frac{\partial E^i}{\partial x^k} - y^j \frac{\partial^2 E^i}{\partial x^j \partial y^k} = 2 \frac{1}{\mu} E^i \mu_k - \frac{1}{\mu} (y\mu) E^i{}_k + (yg)^2 S^i{}_k + \Delta. \quad (\text{C.23})$$

Now it is easy to continue the calculation: we shall use the equality

$$\frac{\partial \widehat{M}}{\partial g} = \frac{\partial M}{\partial g} + \frac{2b^2 q^2}{B^2} \quad (\text{C.24})$$

ensued from (C.5).

We find

$$S^i{}_k = -\frac{1}{2} \left(\frac{B}{b^2} \frac{2}{Ng} \frac{2}{Ng} A_k A^i + (1+gw) \mathcal{H}_k^i + \mathcal{H}_k^i + (1+gw+w^2) \frac{2}{Ng} \frac{2}{Ng} A^i A_k \right) \frac{\partial M}{\partial g}$$

$$\begin{aligned}
& - \left(\frac{B}{b^2} \frac{2}{Ng} \frac{2}{Ng} A_k A^i + (1 + gw) \mathcal{H}^i_k + (1 + gw + w^2) \frac{2}{Ng} \frac{2}{Ng} A^i A_k \right) \frac{b^2 q^2}{B^2} \\
& - \widehat{M} w \left(\frac{1}{2} \mathcal{H}^i_k + \frac{2}{Ng} \frac{2}{Ng} A^i A_k - \frac{2}{Ng} w A_k l^i \right), \tag{C.25}
\end{aligned}$$

where we must insert the derivative

$$\frac{\partial M}{\partial g} = \frac{1}{h^2} \left[\frac{3g}{4} M + \frac{q^2}{B} - \frac{q^2}{B^2} \left(b^2 + \frac{1}{2} g b q \right) \right] \tag{C.26}$$

(see the formulas below (A.82)).

The respective *involutive curvature tensor* R^i_k is constructed according to

$$K^2 R^i_k = 2 \frac{\partial \bar{E}^i}{\partial x^k} - y^j \frac{\partial \bar{E}^i_k}{\partial x^j} - \bar{E}^i_n \bar{E}^n_k + 2 \bar{E}^n \bar{E}^i_{nk} + y^n a_n{}^i{}_{km} y^m, \tag{C.27}$$

where

$$\bar{E}^i = \frac{1}{2} E^i, \quad \bar{E}^i_k = \frac{1}{2} E^i_k, \quad \bar{E}^i_{nk} = \frac{1}{2} E^i_{nk}, \tag{C.28}$$

and $a_n{}^i{}_{km}$ stands for the Riemannian curvature tensor of the associated Riemannian space. The explicit formulas (C.22)–(C.26) must be inserted in (C.27), yielding the following result:

$$\begin{aligned}
& K^2 R^i_k = \frac{1}{\mu} E^i \mu_k - \frac{1}{2\mu} (y\mu) E^i_k \\
& + \frac{1}{2} (yg)^2 \left[-\frac{1}{2} \left(2 \frac{B}{b^2} \frac{2}{Ng} \frac{2}{Ng} A_k A^i + (1 + gw) \mathcal{H}^i_k \right) \frac{\partial M}{\partial g} \right. \\
& - \left(2 \frac{B}{b^2} \frac{2}{Ng} \frac{2}{Ng} A_k A^i + (1 + gw) \mathcal{H}^i_k \right) \frac{b^2 q^2}{B^2} \\
& \left. - \widehat{M} w \left(\frac{1}{2} \mathcal{H}^i_k + \frac{2}{Ng} \frac{2}{Ng} A^i A_k - \frac{2}{Ng} w A_k l^i \right) \right] \\
& + \frac{1}{16} M^2 (yg)^2 \mathcal{H}^k_m + \frac{1}{4} M \widehat{M} (yg)^2 \left(\frac{2}{Ng} \frac{2}{Ng} \frac{B}{b^2} A^k A_m - \frac{1}{4} (1 + gw) \mathcal{H}^k_m \right) \\
& - \frac{1}{4} \widehat{M}^2 (yg)^2 \left(\frac{4}{N^2 g^2} (1 + gw)^2 A^k A_m + \frac{1}{4} \frac{B}{b^2} \mathcal{H}^k_m \right) \\
& - \frac{1}{4} \widehat{M} \frac{2}{Ng} w (yg)^2 \left(2 \frac{2}{Ng} A^k A_m - \widehat{M} (1 + gw) l^k A_m + \frac{2q^2}{B} w l^k A_m + M \frac{Ng}{2} w \mathcal{H}^k_m \right)
\end{aligned}$$

$$+ y^n a_n{}^i{}_{km} y^m. \quad (\text{C.29})$$

In evaluations, we apply the representation

$$\frac{\partial M_i}{\partial g} = -4 \frac{bq^3}{B^2} \frac{2}{KNg} A_i. \quad (\text{C.30})$$

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